Persistent Cohomology and Circle-valued coordinates

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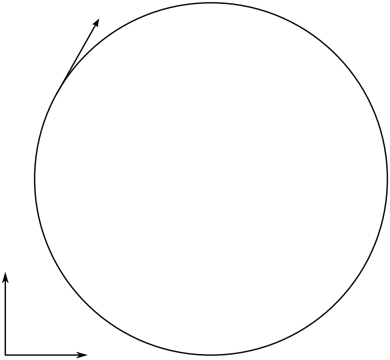
Outline

Finding coordinates

- Overall goal is to understand pointclouds.
- Data comes with coordinates.
  Different coordinate choice might concentrate the intrinsic information.
- We want to find few and very relevant intrinsic coordinates.
  Ideal case: 2d or 3d plots with a clear and relevant geometry.

Problem cases

In order to find few intrinsic coordinates, we want to stick close to the local dimension.
Some shapes take up too many coordinates.



Locally 1-dimensional. Globally 2 coordinates needed to describe all points. The shape doesn't fit in $\mathbb{R}$.

Similar problems arise with sphere and torus.

Suggested fix

Circle-valued coordinates

- Use $S^1 = [0,1]/(0 \sim 1)$ as additional coordinate space
- Fixes the circle
- Fixes the torus
- Occurs naturally:
  - Phase coordinates for waves
  - Angle coordinates for directions
  - Any recurrent phenomenon

Outline

Circle-valued coordinates and cohomology

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Problem remains: how do we find circle-valued coordinates?

Persistent cohomology

- Degree one cohomology equivalent to circle-valued maps
- Persistence picks out relevant features from noise
- Once a feature-rich parameter has been found, we can work in ordinary (non-persistent) cohomology theories

We compute persistent cohomology by adapting the zig zag persistence algorithm to the dual diagram.

From cohomology to circle-valued parametrizations

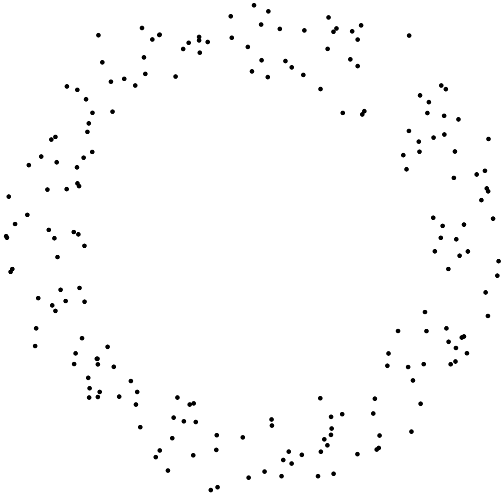
We use the natural isomorphism $H^1(X;\mathbb{Z}) \cong [X, S^1]$

Issues

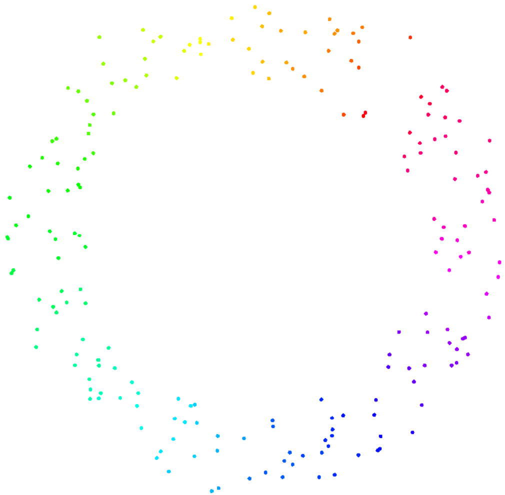
- Easy to compute: $H^1(X;\mathbb{Z}/p)$, with coefficients over a small prime. Linear algebra, coefficients fit inside machine word, division in $O(1)$ by lookup tables.
  Needed for the isomorphism: $H^1(X;\mathbb{Z})$.
  We can, as long as $H^2(X;\mathbb{Z})$ has no p-torsion, lift $H^1(X;\mathbb{Z}/p) \to H^1(X;\mathbb{Z})$.
- The representative chains for $H^1(X;\mathbb{Z})$ yields very non-smooth maps: sends all data points to 0, and wraps the edges in the complex around the target circle.
  We can smooth a cocycle in $C^1(X;\mathbb{Z})$ by moving it to a harmonic cocycle in $C^1(X;\mathbb{R}) \cap C_1(X;\mathbb{R})$ belonging to the same cohomology class in $H^1(X;\mathbb{Z})$.

Outline

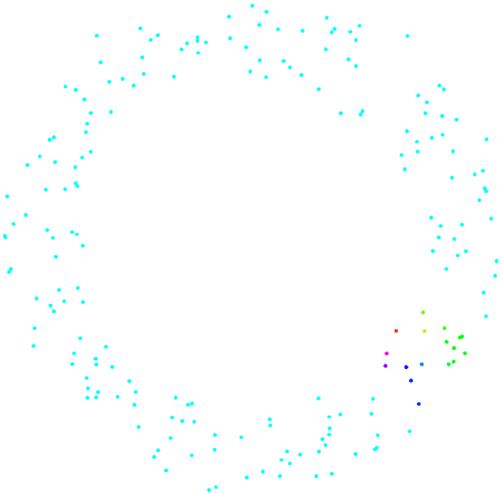
Parametrized circles



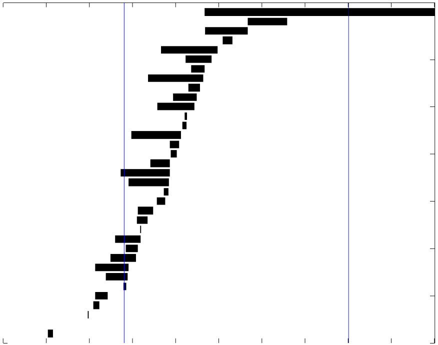
Parametrized circles



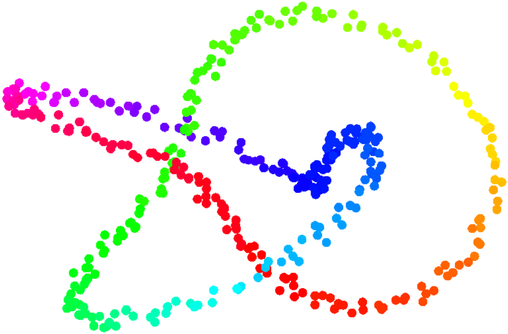
Parametrized circles



Parametrized circles



Knots and links



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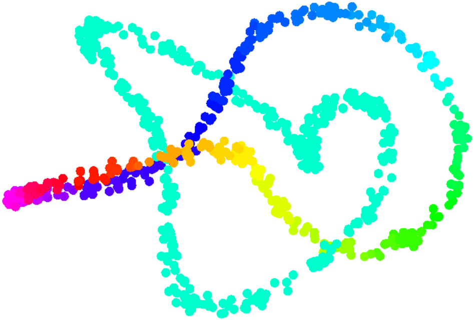

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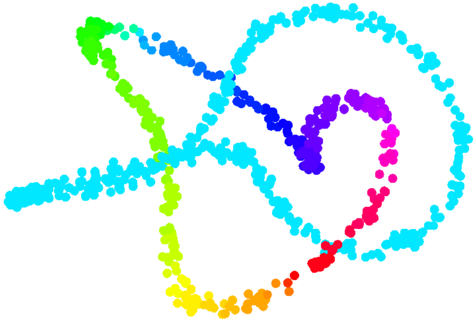
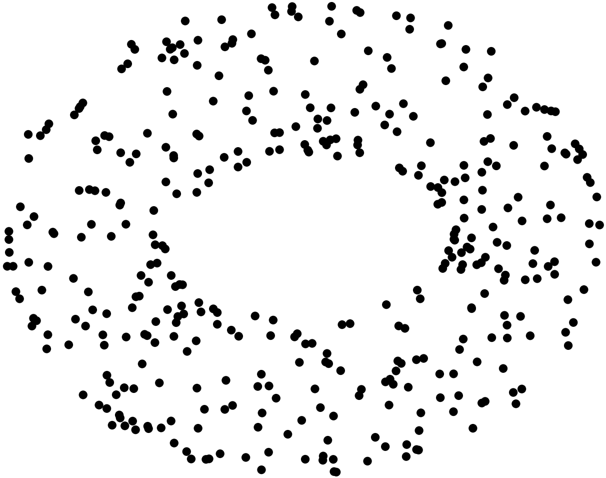


Torus



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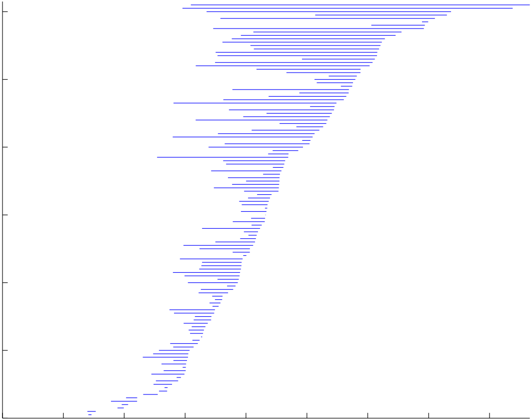

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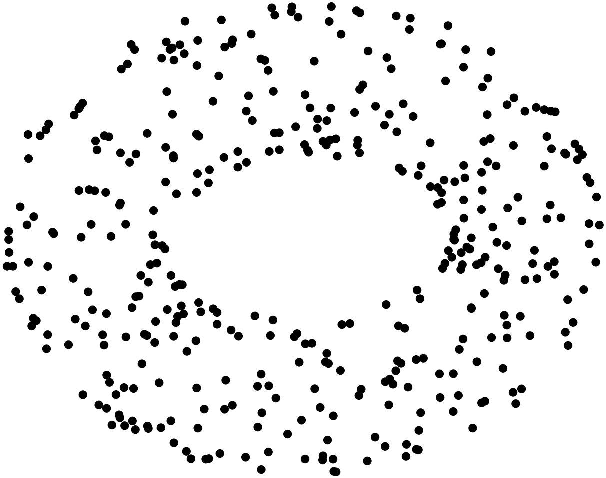

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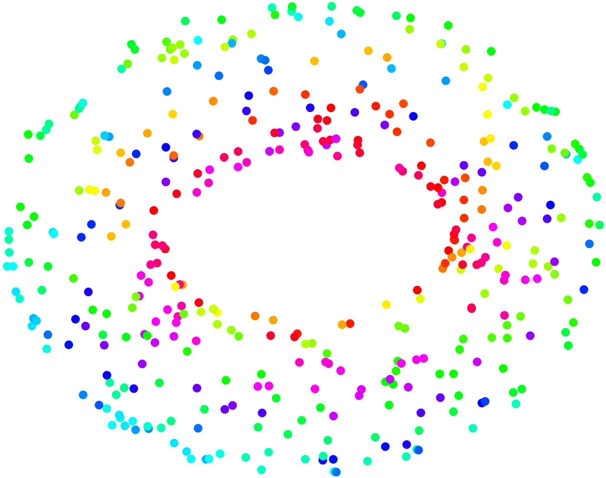

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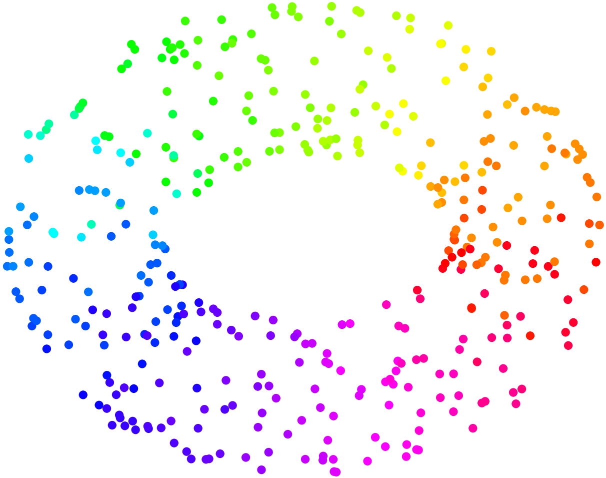

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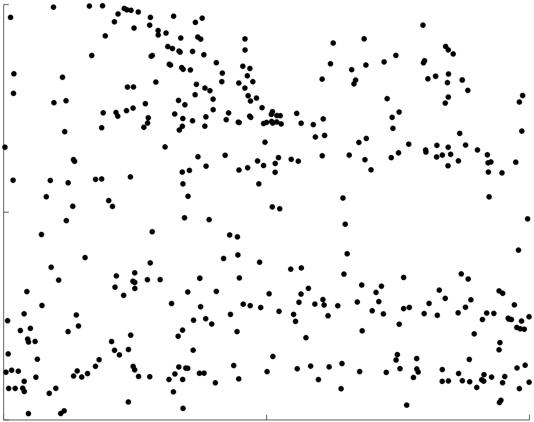
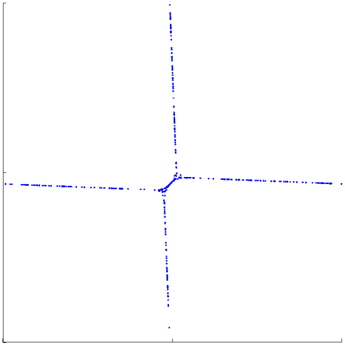


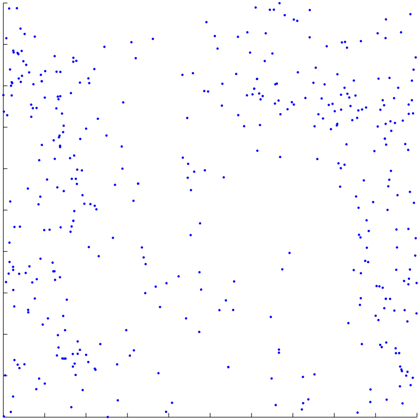
Correlation plot for this torus parametrization

Torus



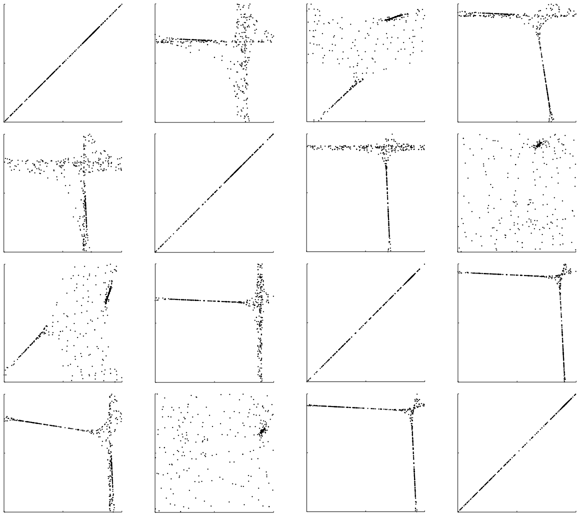
Correlation plot for a wedge of two circles

Torus

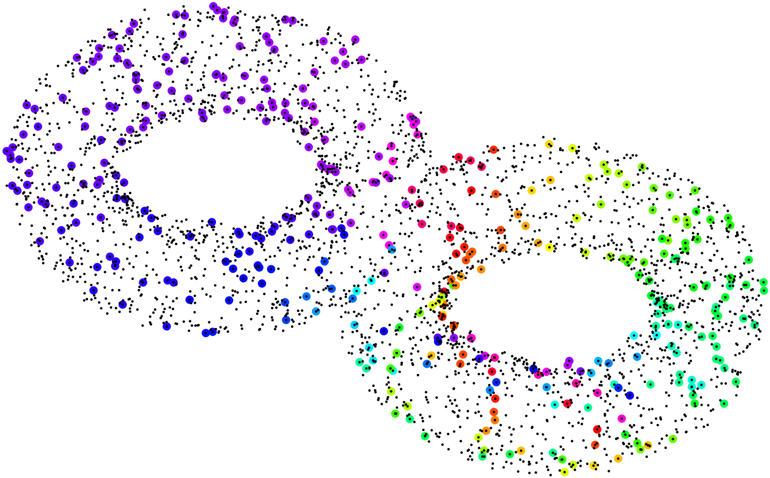


Correlation plot for an elliptic curve in $\mathbb{C}P^2$

Pop quiz



Pop quiz



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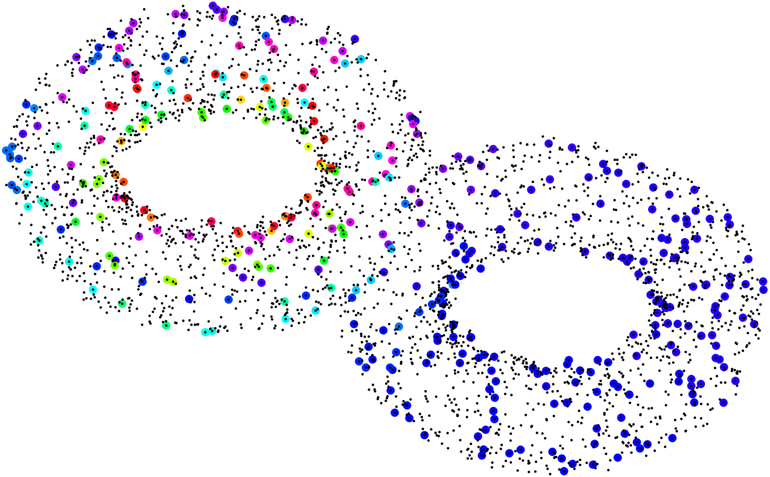

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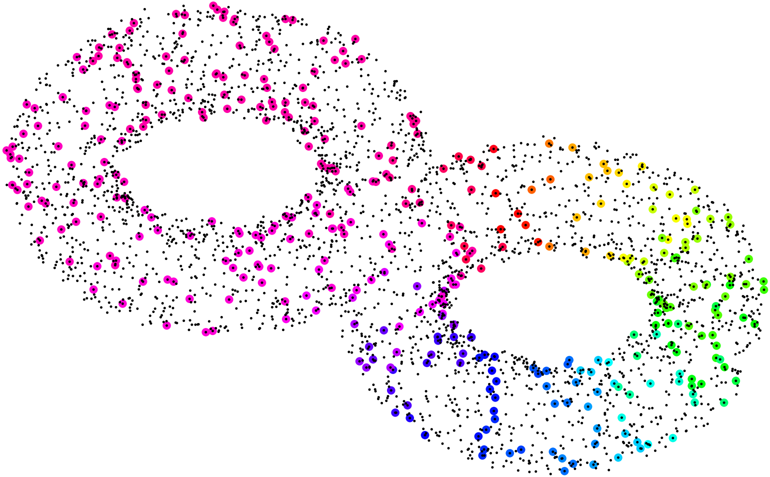

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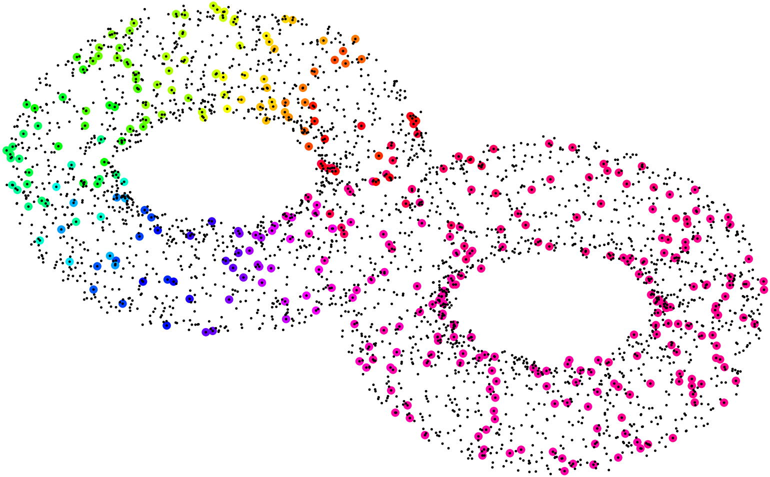


Mumford dataset

Lee-Mumford-Pedersen, *The nonlinear statistics of high-contrast patches in natural images*, International Journal of Computer Vision 54(1/2/3), 83-103, 2003

$4.2 \cdot 10^6$ pixel patches from 4167 calibrated 1020×1532 images.

Each 3×3 pixel patch obviously a vector in $\mathbb{R}^9$. Normalized to constant intensity and to unit euclidean norm.

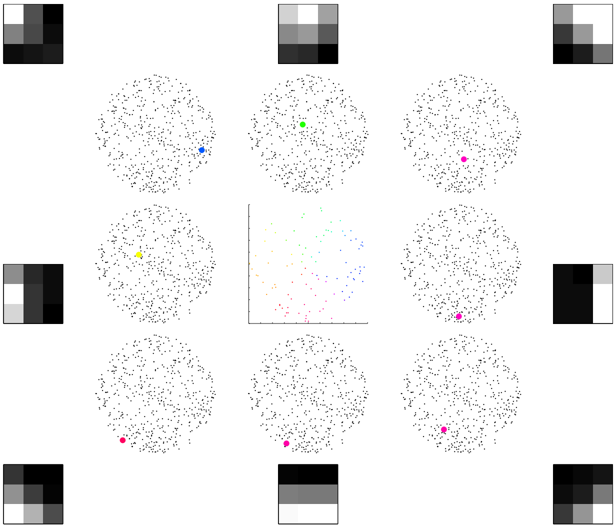
Transformed by a basis choice that highlights geometric features of the dataset itself.

Result lies on the unit 7-sphere in $\mathbb{R}^8$.

Mumford dataset

We use the smoothing procedure developed by Jennifer Kloke. Once smoothed to a circle, we parametrize with persistent cohomology, and can pull the parametrization back to the original data points.

Mumford dataset



Acknowledgements

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